Signatures of the Contravariant Form on Specht Modules for Cyclotomic Hecke Algebras

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Introduction

- Complex Reflection Groups
- Hecke Algebra
- Signatures

Complex Reflection Groups

Let r and n be positive integers. Set of n by n matrices with one nonzero entry in each row and column that is r^{th} root of unity.

Example: Matrix

If ζ is a r^{th} root of unity,

$$\begin{bmatrix} 0 & \zeta & 0 \\ \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^3 \end{bmatrix}$$

is a matrix.

Denoted by $G(r, 1, n)$. Closed under multiplication and inverses.

Representations of Complex Reflection Groups

- For each $M \in G(r, 1, n)$, choose a matrix $g \in GL_j(\mathbb{C}^j)$ and let $\phi(M) = g$
- Satisfies $\phi(M \cdot N) = \phi(M) \cdot \phi(N)$
- Let $V = \mathbb{C}^j$ be a vector space
- For any vector $v \in V$, $\phi(M)(\phi(N)v) = (\phi(M)\phi(N))v$.
- V is called a representation of $G(r, 1, n)$

Partitions and Young Diagrams

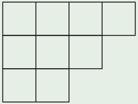
Definition (Partition)

Given an integer l, a *partition* of l is a sequence of integers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ so that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k > 0$ and $\lambda_1 + \lambda_2 + ... + \lambda_k = l$.

Definition (Young Diagram)

- Grid of squares where i^{th} row contains λ_i squares

Example: Young Diagram of $(4, 3, 2)$



Standard Young Tableau and Example

- Fill in boxes of a Young Diagram with numbers 1 to n
- Each row and column is increasing

Example: Standard Young Tableau

Here are examples of Young Tableau:

1	2	3
4	5	
6		

1	3	5
2	4	
6		

Understanding the Complex Reflection Group

- S_n: Set of permutations on n elements
- S_n Sits inside of $G(r, 1, n)$
- S_n is equivalent to $G(1, 1, n)$
- $\mathbb{Z}/r\mathbb{Z}$ also sits inside $G(r, 1, n)$
- $G(r, 1, n)$ is a nontrivial combination of $\mathbb{Z}/r\mathbb{Z}$ and S_n

The Representation Theory of S_n

- The irreducible ("smallest") representations of S_n are characterized by Young Diagrams.
- A basis for each representation is given by Standard Young Tableau.

Example: Representation of S_3

Partition (3):

1	2	3

Partition (2,1):

1	2
3	

1	3
2	

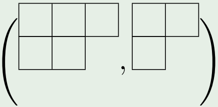
Multi-Young Diagrams

Definition (Multi-Young Diagrams)

Let r and n be fixed integers. A *multi-Young Diagram* of n with r parts is the ordered r-tuple of Young Diagrams $(Y_1, Y_2, ..., Y_r)$, the sum of whose sizes is n.

Example: Multi-Young Diagram

Here is an example of a multi-Young Diagram:



Multi-Tableau

Definition (Multi-Tableau)

If Y is a multi-Young Diagram of size n, a *multi-tableau* of shape Y is given by filling in the numbers from 1 to n in the boxes of Y so that each row and column is increasing.

Example: Multi-Tableau

Here is an example of a multi-Tableau for the same multi-Young Diagram:

$$\left(\begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 7 & 8 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right)$$

Irreducible Representations of $G(r, 1, n)$

- Irreducible Representations of $G(r, 1, n)$ given by multi-Young Diagrams
- Basis for representation given by standard multi-tableau

Representation of $G(r, 1, n)$

Representation of $G(2, 1, 3)$:

$$\left(\begin{array}{|c|c|}\hline & \\ \hline\end{array}\,, \begin{array}{|c|}\hline \\ \hline\end{array}\right)$$

Basis:

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\,, \begin{array}{|c|}\hline 3 \\ \hline\end{array}\right) \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}\,, \begin{array}{|c|}\hline 2 \\ \hline\end{array}\right) \left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}\,, \begin{array}{|c|}\hline 1 \\ \hline\end{array}\right)$$

The Unitary Form

- V is j dimensional representation
- There exists Hermitian form $(-,-) : V \times V \to \mathbb{C}$ so that for all $v, w \in V$,
  - $(v, v) > 0$ for all $v \neq 0$
  - $(av, w) = a(v, w)$
  - $(v, aw) = \overline{a}(v, w)$
  - $(v, w) = \overline{(w, v)}$
  - For all $g \in G(r, 1, n)$, $(\phi(g)v, \phi(g)w) = (v, w)$
- Multi-Tableau orthogonal under this form

Group by Generators and Relations

Definition ($G(r,1,n)$)

The complex reflection group $G(r,1,n)$ is given by the generators $t = s_1$, s_2,..., s_n and relations

$$t^r = 1$$

$$s_i^2 = 1 \text{ for } i \geq 2$$

$$ts_2ts_2 = s_2ts_2t$$

$$s_is_j = s_js_i \text{ for } |i-j| \geq 2$$

and

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } 2 \leq i \leq n-1.$$

Hecke Algebra

Definition (Hecke Algebra)

The *Hecke Algebra* is the algebra generated by $t = s_1$,..., s_n with relations given by

$$(t - u_1)...(t - u_r) = 0$$

$$(s_i + 1)(s_i - q) = 0 \text{ for } i \geq 2$$

$$ts_2ts_2 = s_2ts_2t$$

$$s_is_j = s_js_i \text{ for } |i - j| \geq 2$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } 2 \leq i \leq n - 1$$

where q, u_1,..., u_r are complex numbers on the unit circle but not roots of unity. This algrebra will be denoted by $H_q(r, 1, n)$.

Examples of Other Algebras: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

Irreducible Representations of the Hecke Algebra

Theorem (Irreducible Representations[1])

The set of all finite dimensional irreducible representations of the Hecke Algebra are given by multi-Young diagrams, each with r tableau and total size of n. Moreover, for each multi-Young diagram, a basis is given by the set of all multi-tableau of that shape.

[1] Ariki and Koike

The Contravariant Form and Signature

- Let V be representation of $H_q(r, 1, n)$
- There is Hermitian Form $(-, -) : V \times V \to \mathbb{C}$ on V similar to Unitary Form for $G(r, 1, n)$
- Multi-Tableau are orthogonal under this form
- Problem is $(v, v) > 0$ does not hold for all v
- Let B be basis of multi-tableau of V
- Signature is number of elements $b \in B$ with postive value of (b, b) minus number of elements with negative value of (b, b)
- Invariant of the representation V

Main Theorem

Theorem (Signature)

The signature of a representation of $H_q(r, 1, n)$ is given by

$$\sum_{t_p} \prod_{(i,l)\in D(t_p)} \mathit{sgn}(|u_{f_i} - q^{d_l - d_i} u_{f_l}| - |q - 1|).$$

$D(t_p)$ and Example

- d_i is the column number minus the row number for the tableau i is in. f_i is the number of tableau i is in
- Let t_p be a multi-tableau and let $D(t_p)$ be the set of pairs (i, l) such that $i > l$ and $f_i < f_l$ or i and l are in the same tableau and $d_i < d_l$.

Example: $D(t_p)$

$$\left(\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 6 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right),$$

$D(t_p)$ is $\{(6, 5), (2, 1), (3, 2), (4, 3)\}$.

Consequences and Future Work

- Computation of the Unitary Range
- Same computation for the Cherednik Algebra
- Correspondence between Cherednik Algebra and Hecke Algebra
- Check if signature preserved under correspondence

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References

- Vidya Venkateswaran's paper on Signatures of Cherednik Algebras
- Ariki and Koike's paper on the Hecke Algebra